

MAS204 CALCULUS III

Answers

Section A:

1. [7 marks] Book work(seen in a lecture.)

In index notation, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is

$$\varepsilon_{ijk} A_j \varepsilon_{kmn} B_m C_n,$$

2 marks

and the LHS becomes

$$\varepsilon_{ijk} \varepsilon_{kmn} A_j B_m C_n = \varepsilon_{kij} \varepsilon_{kmn} A_j B_m C_n,$$

1 mark

which the standard two- ε identity changes to

$$(\delta_{im} \delta_{nj} - \delta_{in} \delta_{jm}) A_j B_m C_n,$$

2 marks

simplifying, we get

$$B_i A_j C_j - C_i A_j B_j = B_i (\mathbf{A} \cdot \mathbf{C}) - C_i (\mathbf{A} \cdot \mathbf{B}),$$

1 mark

which we identify as the required RHS

1 mark

2. [10 marks](Similar problems seen.)

(a) Taking the gradient gives the normal to the surface,

$$\mathbf{n} = 2x\mathbf{i} + (2y + \frac{3}{2})\mathbf{j} + (2z - 2)\mathbf{k},$$

2 marks

at the point $(1, 0, 1)$

$$\mathbf{n} = 2\mathbf{i} + \frac{3}{2}\mathbf{j}.$$

1 mark

The tangent plane at the point $(1, 0, 1)$ is $(\mathbf{r} - \mathbf{i} - \mathbf{k}) \cdot \mathbf{n} = 0$ which simplifies to

$$4x + 3y = 4.$$

2 marks

(b) The given sphere can be rewritten as

$$x^2 + (y + \frac{3}{4})^2 + (z - 1)^2 = \frac{25}{16},$$

1 mark

thus the center of the sphere is $(0, -\frac{3}{4}, 1)$.

1 mark

The line which is normal to the tangent plane is parallel to $\mathbf{n}$, so taking into account that this line should pass through the center of sphere we have $x = 2t$, $y = -\frac{3}{4} + \frac{3}{2}t$, $z = 1$, where t is parameter running from $-\infty$ to $+\infty$.

2 marks

(c) When $x = 1$, $t = \frac{1}{2}$ and $y = -\frac{3}{4} + \frac{3}{2} \cdot \frac{1}{2} = 0$, and $z = 1$, hence this line passes through the point $(1, 0, 1)$.

1 mark

3. [6 marks](Similar problems seen.)

(a)

$$\nabla \cdot \mathbf{F}_1 = 2x + 2y + 2z \neq 0,$$

1 mark

$$\nabla \cdot \mathbf{F}_2 = 0 + 0 = 0.$$

1 mark

(b)

$$\nabla \times \mathbf{F}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = 0 \cdot \mathbf{i} - 0 \cdot \mathbf{j} + 0 \cdot \mathbf{k} = \mathbf{0},$$

2 marks

$$\nabla \times \mathbf{F}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = 0 \cdot \mathbf{i} - 0 \cdot \mathbf{j} + (-1 - 1) \cdot \mathbf{k} = -2\mathbf{k} \neq \mathbf{0},$$

2 marks

4. [8 marks](Similar problems seen.)

(a) Using the divergence theorem

$$\oint (x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}) \cdot d\mathbf{S} =$$

2 marks

$$= \int \int \int \nabla \cdot (x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}) dV = \int \int \int (1 + 2 + 3) dV = 6V$$

resulting in 6 times the volume enclosed by the surface.

2 marks

(a) Using Stokes' theorem

$$\oint (x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}) \cdot d\mathbf{r} = \int \int \nabla \times (x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}) \cdot d\mathbf{S} =$$

2 marks

$$= \int \int d\mathbf{S} \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3z \end{vmatrix} = 0,$$

giving zero.

2 marks

5. [8 marks](Similar problems seen.)

(a)

$$\cos(2x + \frac{\pi}{3}) = \cos 2x \cos \frac{\pi}{3} - \sin 2x \sin \frac{\pi}{3} = \frac{1}{2} \cos 2x - \frac{\sqrt{3}}{2} \sin 2x,$$

2 marks

hence

$$a_n = \begin{cases} \frac{1}{2}, & \text{if } n = 2 \\ 0, & \text{if } n \neq 2 \end{cases},$$

1 mark

$$b_n = \begin{cases} -\frac{\sqrt{3}}{2}, & \text{if } n = 2 \\ 0, & \text{if } n \neq 2 \end{cases}.$$

1 mark

(b)

$$\cos^3 x = \cos x \cdot \cos^2 x = \cos x \frac{1}{2}(1 + \cos 2x) = \frac{1}{2} \cos x + \frac{1}{2} \cos x \cdot \cos 2x = \frac{1}{2} \cos x + \frac{1}{8}(e^{ix} + e^{-ix})(e^{2ix} + e^{-2ix}) =$$

$$\frac{1}{2} \cos x + \frac{1}{4} \cos x + \frac{1}{4} \cos 3x = \frac{3}{4} \cos x + \frac{1}{4} \cos 3x,$$

2 marks

hence

$$a_n = \begin{cases} \frac{3}{4}, & \text{if } n = 1 \\ \frac{1}{4}, & \text{if } n = 2 \\ 0, & \text{if } n \neq 1, 2 \end{cases},$$

1 mark

and

$$b_n = 0.$$

1 mark

6. [9 marks](Similar problems seen.)

The line integral will depend on path if $\nabla \times \mathbf{F} \neq \mathbf{0}$ and will not if $\nabla \times \mathbf{F} = \mathbf{0}$.

2 marks

(a)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & z^2 \end{vmatrix} = \mathbf{i}(\frac{\partial z^2}{\partial y} - \frac{\partial(-y)}{\partial z}) - \mathbf{j}(\frac{\partial z^2}{\partial x} - \frac{\partial x}{\partial z}) + \mathbf{k}(\frac{\partial(-y)}{\partial x} - \frac{\partial x}{\partial y}) = \mathbf{i} \cdot 0 - \mathbf{j} \cdot 0 + \mathbf{k} \cdot 0 = \mathbf{0},$$

so the line integral will not depend on path.

2 marks

(b)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & xy \end{vmatrix} = \mathbf{i}(\frac{\partial(xy)}{\partial y} - \frac{\partial(-x)}{\partial z}) - \mathbf{j}(\frac{\partial(xy)}{\partial x} - \frac{\partial y}{\partial z}) + \mathbf{k}(\frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y}) = \mathbf{i} \cdot x - \mathbf{j} \cdot y - 2\mathbf{k} \neq \mathbf{0},$$

so the line integral will depend on path.

2 marks

(c)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{2}y^2z^2 & xyz^2 & xy^2z \end{vmatrix} = \mathbf{i}(\frac{\partial(xyz^2)}{\partial y} - \frac{\partial(xyz^2)}{\partial z}) - \mathbf{j}(\frac{\partial(xyz^2)}{\partial x} - \frac{\partial(\frac{1}{2}y^2z^2)}{\partial z}) + \mathbf{k}(\frac{\partial(xyz^2)}{\partial x} - \frac{\partial(\frac{1}{2}y^2z^2)}{\partial y}) =$$

$$\mathbf{i} \cdot (2xyz - 2xyz) - \mathbf{j} \cdot (y^2z - y^2z) + \mathbf{k}(yz^2 - yz^2) = \mathbf{0},$$

so the line integral will not depend on path.

3 marks

7. [12 marks](Similar problems seen.)

We can use the Hamiltonian instead of the Lagrangian $L = y'^2 - 2y^2$ because this lagrangian does not depend explicitly on x .

2 marks

$$H = y' \frac{\partial L}{\partial y'} = y' \cdot 2y' - y'^2 + 2y^2 = y'^2 + 2y^2 = \text{const},$$

$$y'^2 = H - 2y^2,$$

2 marks

$$dy = \pm \sqrt{H - 2y^2} dx,$$

$$\int \frac{dy}{\sqrt{H - 2y^2}} = \pm x + C.$$

2 marks

By substituting

$$y = \sqrt{\frac{H}{2}} \sin \phi,$$

we have

$$\sqrt{\frac{H}{2}} \int \frac{\cos \phi d\phi}{\sqrt{H - 2 \cdot \frac{H}{2} \sin^2 \phi}} = \frac{\phi}{\sqrt{2}} = \pm x + C,$$

and

$$y = \sqrt{\frac{H}{2}} \sin(C \pm \sqrt{2}x).$$

3 marks

Boundary conditions: in the first point $(0, 0)$ fixes $C = 0$ and the other boundary condition gives

$$1 = \pm \sqrt{\frac{H}{2}} \sin \frac{\pi}{2} = \pm \sqrt{\frac{H}{2}}$$

.

2 marks

Hence

$$y = \sin \sqrt{2}x.$$

1 mark

Section B:

8. (Similar problems seen.)

[Note the question erroneously omitted the constant term, which works out to be $\pi^2/2$. (If you do a sketch, this can be seen by noting that the two parabolic sections fit together to fill a rectangle of area π by π^3 total integral and the constant term is the average of this over the 2π range.)]

$$a_n = \frac{1}{\pi} \int_{-\pi}^0 (\pi + x)^2 \cos nx dx + \frac{1}{\pi} \int_0^{\pi} (\pi^2 - x^2) \cos nx dx =$$

1 mark

$$= \frac{1}{\pi n} \left[\int_{-\pi}^0 (\pi + x)^2 d \sin nx + \int_0^{\pi} (\pi^2 - x^2) d \sin nx \right] =$$

$$= \frac{1}{\pi n} [(\pi + x)^2 \sin nx|_{-\pi}^0 + (\pi^2 - x^2) \sin nx|_0^{\pi} - 2 \int_{-\pi}^0 (\pi + x) \sin nx dx + 2 \int_0^{\pi} x \sin nx dx] =$$

3 marks

$$= \frac{2}{\pi n^2} \left[\int_{-\pi}^0 (\pi + x) d \cos nx - \int_0^{\pi} x d \cos nx \right] =$$

$$= \frac{2}{\pi n^2} [(\pi + x) \cos nx|_{-\pi}^0 - x \cos nx|_0^{\pi} - \int_{-\pi}^0 \cos nx dx + \int_0^{\pi} \cos nx dx] =$$

3 marks

$$= \frac{2}{\pi n^2} \left[\pi - \pi \cos n\pi - \frac{1}{n} \sin nx|_{-\pi}^0 + \frac{1}{n} \sin nx|_0^{\pi} \right] =$$

$$= \frac{2}{n^2} [1 - (-1)^n] = \frac{4}{n^2} \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}.$$

2 marks

$$b_n = \frac{1}{\pi} \int_{-\pi}^0 (\pi + x)^2 \sin nx dx + \frac{1}{\pi} \int_0^{\pi} (\pi^2 - x^2) \sin nx dx =$$

1 mark

$$= -\frac{1}{\pi n} \left[\int_{-\pi}^0 (\pi + x)^2 d \cos nx + \int_0^{\pi} (\pi^2 - x^2) d \cos nx \right] =$$

$$= -\frac{1}{\pi n} [(\pi + x)^2 \cos nx|_{-\pi}^0 + (\pi^2 - x^2) \cos nx|_0^{\pi} - 2 \int_{-\pi}^0 (\pi + x) \cos nx dx + 2 \int_0^{\pi} x \cos nx dx] =$$

3 marks

$$= \frac{2}{\pi n^2} \left[\int_{-\pi}^0 (\pi + x) d \sin nx - \int_0^{\pi} x d \cos nx \right] =$$

$$= \frac{2}{\pi n^2} [(\pi + x) \sin nx|_{\pi}^0 - x \sin nx|_0^{\pi} - \int_{-\pi}^0 \sin nx dx + \int_0^{\pi} \sin nx dx] =$$

3 marks

$$\begin{aligned} &= \frac{2}{\pi n^3} [\cos nx|_{-\pi}^0 - \cos nx|_0^{\pi}] = \\ &= \frac{4}{n^3} [1 - (-1)^n] = \frac{8}{n^3} \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases} . \end{aligned}$$

2 marks

Hence

$$f(x) = 4 \sum_{n \text{ odd}}^{\infty} \frac{1}{n^2} \left(\cos nx + \frac{2}{\pi n} \sin nx \right) .$$

2 marks

9. (Similar problems seen.)

Orthogonality means $\mathbf{e}_u \cdot \mathbf{e}_v = 0$.

2 marks

Since by definition

$$\mathbf{e}_u = \frac{\frac{\partial \mathbf{r}}{\partial u}}{\left| \frac{\partial \mathbf{r}}{\partial u} \right|},$$
$$\mathbf{e}_v = \frac{\frac{\partial \mathbf{r}}{\partial v}}{\left| \frac{\partial \mathbf{r}}{\partial v} \right|},$$

3 marks

it is enough to show that

$$\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} = 0.$$

2 marks

We start from

$$\mathbf{r} = \cos u \cos v \mathbf{i} + \cos u \sin v \mathbf{j}$$

4 marks

and get

$$\frac{\partial \mathbf{r}}{\partial u} = -\sin u \cos v \mathbf{i} - \sin u \sin v \mathbf{j},$$

3 marks

$$\frac{\partial \mathbf{r}}{\partial v} = -\cos u \sin v \mathbf{i} + \cos u \cos v \mathbf{j},$$

3 marks

and working out the dot product gives zero:

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} &= (-\sin u \cos v \mathbf{i} - \sin u \sin v \mathbf{j}) \cdot (-\cos u \sin v \mathbf{i} + \cos u \cos v \mathbf{j}) = \\ &= \sin u \cos v \cos u \sin v - \sin u \sin v \cos u \cos v = 0. \end{aligned}$$

3 marks

Easy for someone who has understood what basis vectors and orthogonality mean.

10. Book work (seen in a lecture.)

(a) In index notations

$$A = \nabla \cdot \nabla \times \mathbf{F} = \partial_i \varepsilon_{ikn} \partial_k \mathbf{F}_n = \varepsilon_{ikn} \frac{\partial^2 \mathbf{F}_n}{\partial x^i \partial x^k} =$$

3 marks

(By antisymmetry of Levi-Chivita Symbol)

$$= -\varepsilon_{kin} \frac{\partial^2 F_n}{\partial x^i \partial x^k} =$$

3 marks

(By symmetry of second partial derivatives)

$$= -\varepsilon_{kin} \frac{\partial^2 F_n}{\partial x^k \partial x^i} =$$

2 marks

(By replacing indices of summation i and k)

$$= -\varepsilon_{ikn} \frac{\partial^2 F_n}{\partial x^i \partial x^k} = -\nabla \cdot \nabla \times \mathbf{F} = -A,$$

or

$$A = -A, \text{ hence } A = \nabla \cdot \nabla \times \mathbf{F} = 0.$$

2 marks

(b) In index notations

$$\mathbf{B} = \nabla \times \nabla \Phi = \mathbf{B}_i = \varepsilon_{ikn} \partial_k \partial_n \Phi = \varepsilon_{ikn} \frac{\partial^2 \Phi}{\partial x^k \partial x^n} =$$

3 marks

(By antisymmetry of Levi-Chivita Symbol)

$$= -\varepsilon_{ink} \frac{\partial^2 \Phi}{\partial x^k \partial x^n} =$$

3 marks

(By symmetry of second partial derivatives)

$$= -\varepsilon_{ink} \frac{\partial^2 \Phi}{\partial x^n \partial x^k} =$$

(By replacing indices of summation n and k)

2 marks

$$= -\varepsilon_{ikn} \frac{\partial^2 \Phi}{\partial x^k \partial x^n} = -B_i,$$

or

$$B_i = -B_i, \text{ hence } B_i = \nabla \times \nabla \Phi = 0.$$

2 marks

11. (Unseen)

Substituting Φ into Laplace's equation

$$\Delta\Phi = 0,$$

1 mark

we have

$$\begin{aligned} & \frac{\partial^2}{\partial x^2} [a(x^n - y^m) + bx^n y^m + c \frac{x^n}{y^m}] + \frac{\partial^2}{\partial y^2} [a(x^n - y^m) + bx^n y^m + c \frac{x^n}{y^m}] = \\ & = a[n(n-1)x^{n-2} - m(m-1)y^{m-2}] + bx^{n-2}y^{m-2}[n(n-1)y^2 + m(m-1)x^2] + c \frac{x^{n-2}}{y^{m+2}}[n(n-1)y^2 + m(m+1)x^2] = 0, \end{aligned}$$

3 marks.

(a) One can see that there are 5

1 mark

pairs of n and m :

1) $n = m = 0,$

1 mark

2) $n = 0, m = 1,$

1 mark

3) $n = 1, m = 0,$

1 mark

4) $n = m = 1,$

1 mark

5) $n = m = 2,$

2 marks

(b) One can see that there are 4

1 mark

pairs of n and m :

1) $n = m = 0,$

1 mark

2) $n = 0, m = 1,$

1 mark

3) $n = 1, m = 0,$

1 mark

4) $n = m = 1,$

2 marks

(c) One can see that there are 2

1 mark

pairs of n and m :

1) $n = m = 0,$

1 mark

2) $n = 1, m = 0,$

1 mark

12. (Similar problems seen[simplified version].)

We express $F(x, y)$ as

$$F(x, y) = A + Bx + Cy + Dxy,$$

where A, B, C, D are constants.

5 marks

From boundary conditions we have:

$$F(x, 0) = A + Bx = 0,$$

hence $A = 0$ and $B = 0$;

3 marks

$$F(x, 1) = C + Dx = x,$$

hence $C = 0$ and $D = 1$;

3 marks

Then we should check two other boundary conditions:

3 marks

$$F(0, y) = 0, \text{ OK,}$$

2 marks

$$F(1, y) = y, \text{ OK.}$$

2 marks

Finally

$$F(x, y) = 0 + 0 \cdot x + 0 \cdot y + 1 \cdot xy = xy.$$

2 marks

13. (Similar problems seen.)

The time is arclength divided by velocity, hence

$$\int \frac{\sqrt{1+y'^2}}{v(y)} dx = \int \sqrt{y(1+y'^2)} dx.$$

4 marks

The simplest way is to use the Hamiltonian form of the Euler-Lagrange equations, because L does not depend explicitly on x ,

2 marks

i.e., to extremize $\int L dx$ we need

$$y' \frac{\partial L}{\partial y'} - L = H = \text{const.}$$

2 marks

Applying this we get

$$-\sqrt{\frac{y}{1+y'^2}} = H$$

which we can rewrite as

$$y^{-1}(1+y'^2) = A = \text{another const.}$$

4 marks

Then we have

$$dy = dx \sqrt{Ay - 1},$$

3 marks

and

$$Ay = 1 + \left[\frac{A(x+C)}{2} \right]^2,$$

where C is an integration constant.

3 marks

The parabola

$$y = 1 + \left(\frac{x-1}{2} \right)^2$$

corresponds to $A = 1$, $C = -1$ and passes through the points $(1, 1)$ and $(3, 2)$.

2 marks